Gap solitons due to cascading

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It is shown analytically that gap solitons can occur in materials with $\chi^{(2)}$ susceptibility due to cascaded second-order nonlinearities. Families of bright and dark spatial gap solitons are described in the framework of asymptotic expansions that are valid, in particular, for nonzero phase mismatch between the first and second harmonics; effective coefficients of self- and cross-phase modulation are calculated.

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The possibility of observing self-focusing phenomena and the propagation of optical solitons is usually connected with the cubic (Kerr) nonlinearity of $\chi^{(3)}$ materials, however it has already been shown theoretically and experimentally that a large nonlinearity-induced phase shift [1,2], self-focusing and self-diffraction [3–5] may be achieved through cascaded $\chi^{(2)} : \chi^{(2)}$ processes. The possible existence of such phenomena were mentioned much earlier (see Ref. [6] as an example) but only recently has it been demonstrated that many of the nonlinear effects predicted for cubic media (or, more generally, Kerr-like materials with the intensity-dependent refractive index) may be expected through cascading, including spatial solitons of quadratic nonlinearities [7–10]. As a matter of fact, in the limit of large (nonlinear) phase mismatch between the first- and second-harmonic field components the soliton dynamics governed by second-order nonlinear effects may be analyzed in the framework of the standard nonlinear Schrödinger (NLS) equation with the effective $\chi^{(3)}_{eff} \sim [\chi^{(2)}]^2$, where $\chi^{(2)}$ stands for the components of the second-order nonlinear susceptibility tensor (see, e.g., [7,9,10]). Such a reduction to an effective NLS equation for quadratic nonlinearities may be justified by a simple asymptotic technique and it is well understood in other models (see, e.g., [11,12]). This allows us to expect that, in spite of different physics, there will be no principal difference (in mathematics) between the effects governed by second- and third-order nonlinearities, at least in the limit of large phase mismatch. Of course, this is probably not true for the opposite case (small or zero phase mismatch) where a strong energy exchange between two harmonics excited in a $\chi^{(2)}$ material may lead to novel nonlinear phenomena (see, e.g., Ref. [10] where a family of two-wave bright solitons has been discovered).

The present paper has a purpose to demonstrate analytically that $\chi^{(2)}$ materials with a spatially periodic linear susceptibility (grating) can support *gap solitons*, self-localized field structures that can occur in nonlinear periodic media in the vicinity of the Bragg resonance. Such gap solitons are known to be possible for materials with Kerr nonlinearity (see the recent review paper [13] and references therein, and also the most recent paper [14]) but here this phenomenon is predicted for nonlinear materials with $\chi^{(2)}$ susceptibility (e.g., noncentrosymmetric crystals), so that one can expect to discover experimentally *spatial gap solitons*, e.g., by using cascaded second-order processes that are known to provide one of the fastest electronic nonlinearities available at the moment. As follows from our analysis, the cascaded second-order effects are also responsible for the effective self- and cross-phase modulation that provide a strong nonlinear coupling between the fields propagating to the right and left.

Following the works on $\chi^{(3)}$ spatial gap solitons in waveguide geometries [15,16], we consider nonlinear electromagnetic waves of the form $\mathcal{E}(x,z;t) = E(x)\exp(ikz-i\omega t)$ propagating along a structure with a periodic dielectric constant $\epsilon(x)$ that possesses second-order (quadratic or $\chi^{(2)}$) nonlinearity. This means that the second harmonic will also be excited in such a medium. Considering interaction of the first and second harmonics we present their fields in the form

$$\mathcal{E}_1(x,z;t) = E_1(x,z)e^{ik_1z-i\omega t},$$
$$\mathcal{E}_2(x,z;t) = E_2(x,z)e^{ik_2z-2i\omega t}, \quad (1)$$

where E_1 and E_2 are assumed to be slowly varying along the propagation direction z and ω is the central frequency of the first harmonic. Substituting Eq. (1) into Maxwell's equations and taking into account the nonlinear coupling through components $\chi^{(2)}_{ijk}$ of the second-order-nonlinearity tensor, we come to the equations,

$$2ik_1\frac{\partial E_1}{\partial z} + \frac{\partial^2 E_1}{\partial x^2} + \Delta k_1^2 E_1 + \chi_1 E_1^* E_2 e^{-i\delta kz} = 0, \quad (2)$$

$$2ik_2\frac{\partial E_2}{\partial z} + \frac{\partial^2 E_2}{\partial x^2} + \Delta k_2^2 E_2 + \chi_2 E_1^2 e^{i\delta kz} = 0, \quad (3)$$

where $\chi_1 \equiv (8\pi\omega^2/c^2)\chi^{(2)}(\omega;2\omega,-\omega)$, $\chi_2 \equiv (16\pi\omega^2/c^2)\chi^{(2)}(2\omega;\omega,\omega)$, δk is the phase mismatch between the first and second harmonics, $\delta k \equiv 2k_1 - k_2$, and $\Delta k_{1,2}^2$ are defined through the expression, $\Delta k_n^2 = -k_n^2 + (\omega_n^2/c^2)\epsilon_n$, where $n = 1,2$ and $\epsilon_n = 1 + 4\pi\chi^{(1)}(\omega_n)$ are linear permittivities calculated for the frequencies $\omega_1 = \omega$ and $\omega_2 = 2\omega$. Equations (2) and (3) are well known in the theory of the second-harmonic generation (see, e.g., [17]) but here, in fact, they are generalized to include the effects produced by diffraction in the transverse direction.

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Now suppose the medium periodic in the x direction (e.g., due to grating) so that $\epsilon_n(x) = \epsilon_{0n} + \tilde{\epsilon}_n(x)$, where ϵ_{0n} is the spatial average and $\tilde{\epsilon}_n(x)$ is periodic with the period $L = \pi/q$. Then we can expand $\tilde{\epsilon}_n(x)$ into a Fourier series and keep only the first harmonics, i.e.,

$$\epsilon_n(x) = \epsilon_{0n} + \epsilon_n^{(m)} \cos(2qx). \quad (4)$$

As is well known, under the Bragg condition, i.e., when one half of a wavelength fits exactly into each period of the grating, such a periodic structure leads to a strong reflection, so that in linear theory the wave propagation near the Bragg resonance is forbidden. In the nonlinear case, a strong interaction between the fields propagating to the right and left can create a self-localized structure known as *a gap soliton* [13]. To derive equations of nonlinear theory, we use a coupled-mode analysis and asymptotic expansions, introducing slowly varying amplitudes F_+ and F_-, for the first harmonic, and G_0, G_+, and G_-, for the second harmonic, of the fields propagating to the right and left. So we look for solutions in the form

$$E_1 = \varepsilon\left(F_+ e^{iqx} + F_- e^{-iqx}\right) + \text{c.c.}, \quad (5)$$

$$E_2 = \varepsilon^2\left(G_0 + G_+ e^{2iqx} + G_- e^{-2iqx}\right) e^{i\delta k z} + \text{c.c.}, \quad (6)$$

where c.c. stands for complex conjugation and the (small) parameter ε is introduced to apply properly a technique of asymptotic expansions. It can be shown that, similar to [7,9,10], the asymptotic expansions can be justified, in particular, for large phase mismatch. However, such asymptotic expansions are in fact valid even for small δk provided *a nonlinear phase mismatch* is big enough (see, e.g., discussions in Ref. [10]).

The amplitudes F_+, F_-, G_0, G_+, and G_- in Eqs. (5) and (6) are considered to be slowly varying in x and z, and, to take into account this property, we assume $\partial/\partial x, \partial/\partial z \sim \varepsilon^2$. At last, the coupled-mode analysis, usually used for deriving coupled equations in the theory of gap solitons [13], is based on the idea that the amplitude of the periodic grating is also small and is of the order of the nonlinearity-induced phase shift, so that here we use also the condition $\epsilon_n^{(m)} = \varepsilon^2 a_n$, where $a_n = O(1)$, $n = 1, 2$.

Substituting Eqs. (5) and (6) into Eq. (2) and considering the scaling relations mentioned above, we obtain the following system of equations for F_+ and F_-:

$$2ik_1 \frac{\partial F_+}{\partial z} + 2iq \frac{\partial F_+}{\partial x} + a_1 F_- + \chi_1 (G_0 F_-^* + G_+ F_+^*) = 0, \quad (7)$$

$$2ik_1 \frac{\partial F_-}{\partial z} - 2iq \frac{\partial F_-}{\partial x} + a_1 F_+ + \chi_1 (G_0 F_+^* + G_- F_-^*) = 0, \quad (8)$$

where the wave vectors for both harmonics are fixed by the dispersion relations,

$$k_1^2 = \frac{\omega^2}{c^2}\epsilon_{01} - q^2, \quad k_2^2 = \frac{4\omega^2}{c^2}\epsilon_{02} - 4q^2. \quad (9)$$

It is important to note that in Eqs. (7) and (8) all terms are of the same order, but nonlinear parts involve also the amplitudes of the second harmonic. These additional equations must follow from Eq. (3) which after substitution of Eqs. (5) and (6), are reduced, in the lowest order in ε, to the simple algebraic relations,

$$G_0 = \frac{\chi_2}{(k_2 \delta k - 2q^2)} F_+ F_-, \quad (10)$$

$$G_+ = \frac{\chi_2}{2k_2 \delta k} F_+^2, \quad G_- = \frac{\chi_2}{2k_2 \delta k} F_-^2. \quad (11)$$

Using Eqs. (10) and (11) we transform Eqs. (7) and (8) to the system of coupled equations for the amplitudes F_+ and F_- of the fields propagating to the right and left,

$$i\frac{\partial F_+}{\partial z} + i\delta \frac{\partial F_+}{\partial x} + \Delta F_- + \Gamma_S |F_+|^2 F_+ + \Gamma_X |F_-|^2 F_+ = 0, \quad (12)$$

$$\frac{\partial F_-}{\partial z} - i\delta \frac{\partial F_-}{\partial x} + \Delta F_+ + \Gamma_S |F_-|^2 F_- + \Gamma_X |F_+|^2 F_- = 0, \quad (13)$$

where $\delta \equiv q/k_1$, $\Delta \equiv a_1/2k_1$, and the coefficients Γ_S and Γ_X,

$$\Gamma_S = \frac{\chi_1 \chi_2}{4k_1 k_2 \delta k}, \quad \Gamma_X = \frac{\chi_1 \chi_2}{2k_1 (k_2 \delta k - 2q^2)}, \quad (14)$$

describe the effective self- and cross-phase modulation due to cascading. Equations (12) and (13) look similar to the basic nonlinear equations of the theory of gap solitons in a Kerr medium [13], however, for a pure cubic nonlinearity $\Gamma_X = 2\Gamma_S$. The same relation holds here between Γ_S and Γ_X in the limit of large phase mismatch, $\delta k \gg q^2/k_2$.

Substituting the solutions $F_+, F_- \sim \exp(iQx - i\beta z)$ into the linearized version of Eqs. (12) and (13), we come to the conclusion that in the vicinity of the Bragg resonance there is the spectrum gap, $|\beta| < \Delta$. To find stationary solutions of *nonlinear* equations (12), (13), we use the transformation (cf. [18])

$$F_+ = (u + iv)e^{-i\beta z}, \quad F_- = (u - iv)e^{-i\beta z}, \quad (15)$$

where $u(x)$ and $v(x)$ are real functions, and obtain the system of ordinary differential equations for u and v,

$$\frac{dv}{dx} = \Delta_+ u + \gamma(u^2 + v^2)u, \qquad \frac{du}{dx} = -\Delta_- v - \gamma(u^2 + v^2)v, \quad (16)$$

where $\Delta_\pm \equiv (\beta \pm \Delta)/\delta$ and $\gamma \equiv (\Gamma_S + \Gamma_X)/\delta$. Equations (16) may be treated as a Hamiltonian system, with u as a generalized momentum and v as a generalized coordinate, and the conserved energy (Hamiltonian),

$$E = \tfrac{1}{2}(\Delta_+ u^2 + \Delta_- v^2) + \tfrac{1}{4}\gamma(u^2 + v^2)^2. \quad (17)$$

Then different types of stationary solutions are characterized by different values of E. We are interested here in spatially localized solutions which on the phase plane (u, v) correspond to separatrix trajectories. Not restricting generality of our analysis we take $\gamma > 0$, so that for positive Δ_+ and Δ_- there are no separatrix trajectories, i.e., localized solutions of Eqs. (16) are absent. However, when $\Delta_+ > 0$ and $\Delta_- < 0$, i.e., when β is selected just within the gap of the linear spectrum, $|\beta| < \Delta$, the separatrix trajectory on the plane (u, v) starts from the unstable (saddle) critical point $(0,0)$ and returns to it, describing a spatially localized mode with even $v(x)$ and odd $u(x)$ functions, which corresponds to *a bright gap soliton.*

Introducing the auxiliary function $g = u/v$, we reduce the system (16), with the help of Eq. (17), to the equation

$$\left(\frac{dg}{dx}\right)^2 = (\Delta_- + \Delta_+ g^2)^2 + 4\gamma E(1+g^2)^2, \qquad (18)$$

which may be easily integrated (see the similar approach used in [19,20]). For the separatrix trajectory starting from and coming to the point $(0,0)$ we determine $E = 0$, so that the solution is

$$g(x) = \pm\sqrt{\frac{|\Delta_-|}{\Delta_+}}\tanh\xi, \quad \xi \equiv x\sqrt{\Delta_+|\Delta_-|}, \qquad (19)$$

and, therefore,

$$v^2(x) = \frac{2|\Delta_-|\Delta_+^2\cosh^2\xi}{\gamma(\Delta_+\cosh^2\xi + |\Delta_-|\sinh^2\xi)^2},$$
$$u(x) = g(x)v(x). \qquad (20)$$

The solution (15), (19), and (20) presents another form of the well-known bright gap soliton (see, e.g., [13] and references therein).

If both Δ_- and Δ_+ are negative, the critical point $(0,0)$ on the phase plane (u, v) becomes *stable*, and bright gap solitons do not exist. This corresponds to the case when the propagation constant β is selected *below* the gap, $\beta < -\Delta$. In this case we find *dark gap solitons* similar to those discovered for diatomic lattices [19] (see also [21]). The separatrix trajectory corresponding to such a dark soliton connects two unstable (saddle) critical points $(u_0, 0)$ and $(-u_0, 0)$, where $u_0^2 = |\Delta_+|/\gamma$. Calculating the value of E for this trajectory, $-\Delta_+^2/4\gamma$, we find the solution $g(x) = \pm A\sinh(Bx)$, where $A^2 = (|\Delta_-| - |\Delta_+|)/2|\Delta_+|$ and $B^2 = 2|\Delta_+|(|\Delta_-| - |\Delta_+|)$, which together with Eq. (17) allows to find u and v,

$$v^2(x) = \frac{1}{\gamma(1+g^2)^2}\Big[(|\Delta_-| + |\Delta_+|g^2) \pm\sqrt{(|\Delta_-| + |\Delta_+|g^2)^2 + 4\gamma E(1+g^2)^2}\Big], \qquad (21)$$

$u(x) = g(x)v(x)$. In that case the even function $v(x)$ is again spatially localized but the odd function $u(x)$ has nonvanishing asymptotics, so that the envelopes F_+ and F_- describe a localized wave on a background with the intensity $u_0^2 = |\Delta_+|/\gamma$. As follows from Eq. (21), there are two types of dark gap solitons, one with a lower intensity and the second one, with a larger intensity relative to the background intensity. However, the stability of dark solitons remains to be investigated.

In conclusion, we have predicted analytically that gap solitons can be observed in materials with second-order nonlinearities through the cascaded $\chi^{(2)} : \chi^{(2)}$ processes. Using the asymptotic technique, we have derived the system of coupled-mode equations for the slowly varying amplitudes of the fields propagating to the right and left, and we have determined coefficients of the effective self- and cross-phase modulation that appear due to cascading. We have also suggested a simple (and straightforward) way to find stationary localized solutions corresponding to bright and dark gap solitons. The results obtained allow us to conclude that $\chi^{(2)}$ materials may be well suited for the experimental study of (spatial or temporal) gap solitons and other nonlinearity-induced effects in periodic media.

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